1. Fie x un numar real. Comparati numerele:

$a = 3x^4$ si $b = 8x^3 - 16$

Rezolvare:

1. Stabilim semnul diferentei.

$a - b = 3x^4 - (8x^3 - 16)$

$$3x^4 - 8x^3 + 16 = 3x^4 - 6x^3 - 2x^3 + 16 = 3x^3(x-2) - 2(x^3 - 8)$$
$$= (x-2)(3x^3 - 2x^2 - 4x - 8)$$
$$= (x-2)(3x^3 - 6x^2 + 4x^2 - 8x + 4x - 8) = (x-2)^2(3x^2 + 4x + 4)$$

$3x^2 + 4x + 4 > 0, (\forall)\ x \in R$ deoarece $3x^2 + 4x + 4 = 3(x + \frac{2}{3})^2 + \frac{8}{3}$

$\Rightarrow a - b \geq 0$

$a > b$ pentru $R \backslash \{2\}$

$a = b$ pentru $x = 2$

2. Fie numerela reale a, b, c>0. Aratati ca:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a})$$

Daca si numai daca a=b=c.

Rezolvare:

Inmmultind inegalitatea din enunt cu 2 avem:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{4}{a+b} - \frac{4}{b+c} - \frac{4}{c+a} = 0$$

$$\left(\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b}\right) + \left(\frac{1}{b} + \frac{1}{c} - \frac{4}{b+c}\right) + \left(\frac{1}{a} + \frac{1}{c} - \frac{4}{a+c}\right) = 0$$

$$\frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} = \frac{y(x+y) + x(x+y) - 4xy}{xy(x+y)} = \frac{xy + y^2 + x^2 + xy - 4xy}{xy(x+y)}$$
$$= \frac{x^2 - 2xy + y^2}{xy(x+y)} = \frac{(x-y)^2}{xy(x+y)}$$

$\Rightarrow \frac{(a-b)^2}{ab(a+b)} + \frac{(b-c)^2}{bc(b+c)} + \frac{(c-a)^2}{ca(c+a)} = 0$

$(a-b)^2 \geq 0, (b-c)^2 \geq 0, (c-a)^2 \geq 0, (\forall)$ a, b, c $\in R$

$a, b, c > 0 \Rightarrow ab, bc, ca > 0$ si $a+b, b+c, c+a > 0$

$\Rightarrow$ numitorii fractiilor $> 0, (\forall) a, b, c > 0$

$\Rightarrow$

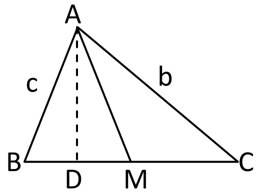
$\Rightarrow \frac{(a-b)^2}{ab(a+b)} = \frac{(b-c)^2}{bc(b+c)} = \frac{(c-a)^2}{ca(c+a)} = 0$

$$\Rightarrow \left\{ \begin{array}{l} (a-b)^2 = 0 \Rightarrow a = b \\ (b-c)^2 = 0 \Rightarrow b = c \\ (c-a)^2 = 0 \Rightarrow c = a \end{array} \right\} \Rightarrow a = b = c$$

3. In triunghiul ABC construim bisectoarea [AD si mediana [AM], D, M ∈ (BC). Daca $\frac{A_{\Delta ADM}}{A_{\Delta ABC}}=\frac{1}{k}$

   $k \in N^*$. Aflati k pentru care $\frac{AC}{AB}$ este numar natural.

Rezolvare:



Notam AB=c, AC=b, BC=a. Deoarece triunghiurile ADM si ABC au aceeasi inaltime avem $\frac{A_{\Delta ADM}}{A_{\Delta ABC}}=\frac{DM}{BC} \Rightarrow \frac{DM}{BC}=\frac{1}{K}$

Din teorema bisectoarei $\frac{BD}{DC}=\frac{c}{b}$ deducem $BD=\frac{ac}{b+c}$, de unde $DM=\frac{a}{2}-\frac{ac}{b+c}=\frac{a(b-c)}{2(b+c)}$

$$\left.\begin{array}{l} \frac{DM}{BC}=\frac{1}{k} \\ DM=\frac{a(b-c)}{2(b+c)} \end{array}\right\} \Rightarrow \frac{b-c}{2(b+c)}=\frac{1}{k}$$

$$\Rightarrow \frac{2b}{b+c}=\frac{2+k}{k} \Rightarrow \frac{2b}{c}=\frac{2+k}{\frac{k}{2}-1} \Rightarrow \frac{2b}{c}=\frac{2(2+k)}{k-2} \Rightarrow \frac{b}{c}=\frac{k+2}{k-2} \Rightarrow \frac{b}{c}=1+\frac{4}{k-2} \Rightarrow \left.\begin{array}{l} \frac{AC}{AB}=1+\frac{4}{k-2} \\ \frac{AC}{AB} \in N \end{array}\right\} \Rightarrow$$

$\Rightarrow k-2 \mid 4 \Rightarrow k \in \{3,4,6\}$

4. Aratati ca oricare ar fi x,y,z numere reale, au loc inegalitatile:

   a)$x^2+y^2+z^2 \geq xy+xz+yz$

   b)$x^4+y^4+1 \geq xy(x+y+1)$

Rezolvare:

a) $x^2+y^2+z^2 \geq xy+xz+yz \Leftrightarrow (x-y)^2+(y-z)^2+(x-z)^2 \geq 0$
b) $x^4+y^4+1 \geq x^2y^2+x^2+y^2 \geq x^2y+xy^2+xy=xy(x+y+1)$

5. a)Demonstrati ca punctual de intersectie al bisectoarelor unghiurilor unui dreptunghi sunt varfurile unui patrat.

   b)Prin varful C al patratului ABCD se construieste o dreapta care intersecteaza (AB si (AD in M, respective N. Demonstrati ca $\frac{1}{AM}+\frac{1}{AN}$ este o constanta.

   Rezolvare:

a. Se dem. ca intersectie al bisectoarelor unghiurilor unui dreptunghi sunt varfurile unui deptunghi cu doua laturi congruente

b. $\frac{1}{AM}+\frac{1}{AN}=\frac{1}{l}$ unde l=latura patratului